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The Variational Treatment of Thick Interacting Inductive Irises

TULLIO E. ROZZI

Abstract—The problem of two thick interacting inductive irises in waveguide is treated with a variational approach.

Using the appropriate Green's functions in the continuity equations of the transverse magnetic fields yields two coupled integral equations for the magnetic currents on the apertures. Solving one equation by Fourier expansion and introducing in the remaining equation, a variational expression for the driving-point admittance is obtained. This is treated with a Rayleigh–Ritz procedure and matrix methods, avoiding the explicit computation of field amplitudes.

The analysis is carried out in terms of an eigenmode expansion, as well as in terms of an expansion à la Schwinger on the aperture and the features of the two methods are contrasted.

In spite of its somewhat greater mathematical complexity, the latter generally provides a superior solution for a given order of the trial field.

In both cases the solutions are very accurate, uniformly convergent to their common limit value, and require manipulations with small-order matrices only. The agreement with the experiment is excellent.

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I. Introduction

The problem of the inductive iris in waveguide, one of the geometrically simplest and most commonly used configurations, admits, nonetheless, no general analytical solution. On the other hand, the variational approach to this problem can be developed analytically to such an extent as to yield answers that can be as accurate as prescribed and, in the quasistatic limit, can even be cast in closed form.

A good discussion of the variational solution of the problems of the infinitely thin iris under TE_{10} incidence, originally due to Schwinger, can be found in a standard textbook by Collin [1].

This solution has been extended to the general case of arbitrary TE_{n0} incidence by Palais [2]. The same author has also discussed the effect of interaction between infinitely thin irises [3]. The thick (isolated) iris is also treated in [1].

The simultaneous presence of finite thickness and interaction between neighboring irises has been discussed in a previous contribution [4]. This utilizes Schwinger's form for the susceptance of an infinitely thin iris, parallel transmission lines for the iris eigenmodes, and ideal transformers at the interfaces. The transformer ratios are expressible as linear combinations of the amplitudes of two, *independently assumed*, trial fields at the interfaces. Field amplitudes are then determined by searching for the stationary point of the driving-point admittance of the even/odd-mode equivalent network.

It is well known in fluid dynamics that flow occurs without turbulence at the junction of two ducts having different cross sections when these can be mathematically related by means of conformal mapping. In the boundary-value problem posed by the iris, a somewhat analogous situation arises if one utilizes an orthogonal set on the iris aperture introduced by Schwinger [1], [5]. These and the x derivatives of the guide eigenmodes satisfy compatible boundary conditions on the aperture edge, and the latter set can be expressed as a finite expansion in terms of the first set. The transformation matrix from the one set to the other is lower triangular and well conditioned.

In order to take account of propagation in the iris (thickness effect), it is, however, necessary to transform back to the representation of the iris eigenmodes, i.e., in network terms, ideal transformers must be introduced at the interfaces. The actual aperture eigenmodes, on the other hand, are the natural set for describing propagation on the thick iris, but their boundary conditions are incompatible with those of the guide modes. These considerations apply to the general case of transverse discontinuities of the aperture type. The aperture eigenmodes are just a choice of an orthonormal set on the aperture and as such, in general, not an optimum one, from the point of view of "bringing the wave" into the aperture. The question arises, therefore, whether an expansion à la Schwinger, in spite of the more complex representation associated with it in the case of finite thickness, provides a more efficient solution.

The purpose of this paper is to present the complete variational solution of the problem of thick, interacting, inductive irises starting from a rigorous integral-equation formulation.

This results in a pair of coupled integral equations: the first one for the driving-point admittance of the even/odd-mode equivalent circuit and the second one relating the two magnetic current distributions occurring at the two interfaces of the iris.

When the second equation is solved by Fourier expansion and the resulting linear relationship between the two distributions is introduced into the first equation, the latter can be reduced to a modified symmetric Rayleigh quotient containing the Fourier amplitudes of only one of the distributions.

The stationary value of the quotient is then found by algebraic methods avoiding the explicit calculation of modal amplitudes. The analysis is carried out by means of an iris eigenmode expansion (Section II), as well as by an expansion in Schwinger's functions (Section III), and the characteristics of the two solutions are investigated. In Section IV, numerical results obtained with both methods are compared with those obtained with the previous method and with the experiment. Only TE_{10} incidence will be explicitly considered. The following results can be extended to general TE_{n0} incidence, as shown in [2] for the infinitely thin iris. The case of the capacitive iris can be treated on parallel lines.

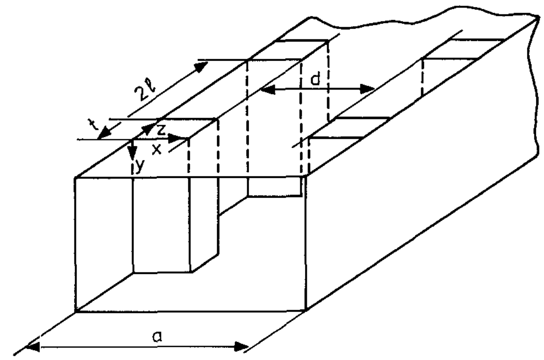

Fig. 1. Geometry.

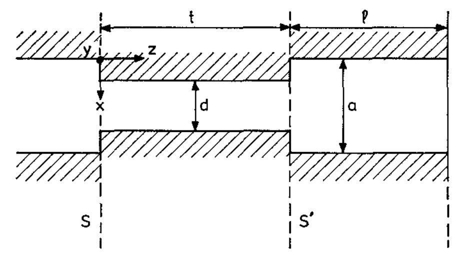

Fig. 2. Half structure for even/odd excitation.

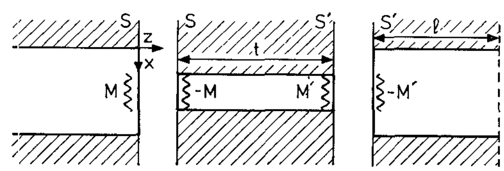

Fig. 3. Equivalent boundary-value problem.

II. The Integral-Equation Formulation and the Eigenmode Expansion

The geometry under study is depicted in Fig. 1. Making use of the mirror symmetry with respect to the plane $z=t+l$, we shall split the problem into an even and odd part by locating a magnetic and an electric wall, respectively, at the plane of symmetry (Fig. 2).

We can solve the field problem separately in each region of the left-half space ($z=t+l$) by applying a well-known field equivalence principle [6].

The two sides of the aperture S at $z=0$ can be considered closed by an electric wall superimposed by a magnetic current distribution $\boldsymbol{M}=\hat{z}\times\boldsymbol{E}$ and $-\boldsymbol{M}$, respectively, $\hat{z}$ being the unit normal in the positive z direction and $\boldsymbol{E}$ the electric field at the aperture location. Similarly, the aperture S' can be closed by an electric wall and the magnetic current distribution $\boldsymbol{M}'=\hat{z}\times\boldsymbol{E}'$ on the left-hand side and $-\boldsymbol{M}'$ on the right-hand side. This equivalence is depicted in Fig. 3. The total (x-

directed) transverse magnetic field in region 1, e.g., incident field plus field radiated by the magnetic current at $z=0$, can be expressed as

$$H_1(r) = I\varphi_1(x)\cos\beta z + j\int_S M(x')\cdot G_1(r;x')\cdot dx'$$

$$\varphi_1(x) = \sqrt{\frac{2}{a}}\sin\frac{\pi x}{a} \tag{1}$$

where I is the amplitude of the current carried by the fundamental mode (TE_{10}). The symmetry of the discontinuity in the y direction causes only TE_{n0} modes to be excited. The reactance dyadic in region 1, at the location $z'=0$, is

$$G_1(x,z;x') = \sum_{n>1}^{\infty}\frac{\Gamma_n}{a\beta}\cdot\sin\frac{n\pi}{a}x\sin\frac{n\pi}{a}x'\cdot\cosh\Gamma_n z. \tag{2}$$

If the iris is symmetric, the sum is restricted to n odd. All characteristic admittances have been normalized to that of the fundamental mode (Γ_n is real for $n>1$ and $\Gamma_1=j\beta$).

In region 2, the field is that radiated by the magnetic currents $-M$ and M' in a resonator of length t and width d. Equivalently, the total field in region 2 is the superposition of the fields present in the two semi-infinite regions $t\geq z\geq -\infty$ and $0\leq z\leq +\infty$.

$$H_1 = H_2 + H_2', \qquad \text{on } S \tag{7a}$$

and

$$H_2 + H_2' = H_3, \qquad \text{on } S'. \tag{7b}$$

Inserting in the above expressions the field expansions (1), (3), (5), and (6), we have

$$Ih(x,0) = -j\left\{\int\int [G_1(x,0;x') + G_2(x,0;x')]M(x')\,dx' + \int M'(x')\cdot G_2'(x,0;x')\,dx'\right\} \tag{8a}$$

and

$$\int_S M(x')G_2(x,t;x')\,dx' - \int_{S'} M'(x')G_2'(x,t;x')dx' = \int_{S'} M'(x')\cdot G_3(x,t;x')\cdot dx'. \tag{8b}$$

This is a system of two coupled integral equations for the yet unknown functions $M(x)$ and $M'(x)$.

From (8a) we can easily derive the following variational expression for the driving-point admittance of the one-port [1]:

$$\frac{2\times 2V\times I}{(2V)^2} = \frac{I}{V} = -jb_0 = \frac{-2j\int\int\{M(x)[G_1(x,0;x')+G_2(x,0;x')M(x') - M'(x)\cdot G_2'(x,0;x')M(x')\}\cdot dx\cdot dx'}{\left[\int M(x)\varphi_1(x)\,dx\right]^2}. \tag{9}$$

The transverse field H_2 radiated by $-M$ in region 2 is obtained by a formula analogous to (1) by setting $I=0$, i.e.,

$$H_2(x,z) = -j\int_S M(x')\cdot G_2(x,z;x')\cdot dx' \tag{3}$$

where the appropriate Green's function now is

$$G_2(x,z;x') = \sum_{m=1}^{\infty}\frac{\gamma_m}{d\beta}\sin\frac{m\pi}{d}\left(x-\frac{a-d}{2}\right)\sin\frac{m\pi}{d}\left(x'-\frac{a-d}{2}\right)\cdot\frac{\cosh\gamma_m(z-t)}{\sinh\gamma_m t}. \tag{4}$$

Similarly, the field H_2' radiated by M' in region 2 is expressed as

$$H_2'(x,z) = j\int_{S'} M'(x')\cdot G_2'(x,z;x')\cdot dx'. \tag{5}$$

The transverse magnetic field in region 3 is given by

$$H_3(x,z) = j\int_{S'} M'(x')\cdot G_3(x,z;x')\cdot dx'. \tag{6}$$

All transverse magnetic fields have now been written in terms of two unknown magnetic current distributions: $\boldsymbol{M}=\hat{z}\times\boldsymbol{E}$ located at S and $\boldsymbol{M}'=\hat{z}\times\boldsymbol{E}'$ located at S' (e.g., in terms of the electric field distributions at the interfaces). Two integral equations for these quantities are obtained by applying the conditions of continuity of the transverse magnetic fields at S and S'. These are

Separating variables and introducing the explicit form of the Green's functions, (8b) is rewritten as

$$\sum_{m=1}^{\infty}\frac{\gamma_m}{\beta}\cdot\operatorname{csch}\gamma_m t\cdot\left(\int\psi_m(x')M(x')\,dx'\right)\psi_m(x) = \sum_{m=1}^{\infty}\frac{\gamma_m}{\beta}\coth\gamma_m t\left(\int\psi(x')M'(x')\,dx'\right)\psi_m(x) + \sum_{m=1}^{\infty}\frac{\Gamma_n'}{\beta}\left(\int\varphi_n(x')\cdot M'(x')\,dx'\right)\phi_n(x) \tag{10}$$

where

$$\phi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x$$

$$\psi_m(x) = \sin\frac{m\pi}{d}\left(x-\frac{a-d}{2}\right)$$

$$\Gamma_n' = \Gamma_n\begin{cases}\tanh\Gamma_n l\\ \coth\Gamma_n l.\end{cases}$$

We shall limit our attention to the symmetric iris, since the asymmetric case presents no new feature, and therefore we shall take for $M(x)$ and $M'(x)$ an expansion in terms of the aperture eigenmodes of odd order:

$$M(x) = \sum_{m=1}^{N}\lambda_m\psi_m(x) \tag{11a}$$

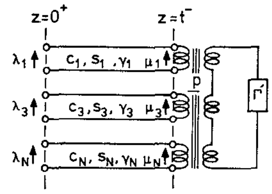

Fig. 4. Equivalent network representation of (16).

$$M'(x) = \sum_{m=1}^{N} \mu_m \psi_m(x). \tag{11b}$$

We introduce now in the right-hand side of (10) the expansion

$$\phi_n(x) = \frac{2}{\sqrt{ad}} \sum_{m=1}^{\infty} P_{mn} \psi_m(x) \tag{12}$$

with

$$P_{mn} = \frac{2m\pi}{d} \frac{\sin \frac{n\pi}{2a}(a-d)}{\left(\frac{m\pi}{d}\right)^2 - \left(\frac{n\pi}{a}\right)^2} \tag{13}$$

and we equate Fourier amplitudes. Some algebraic manipulation yields the following linear relationship between the amplitude vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$:

$$\boldsymbol{\lambda} = (c + s\gamma^{-1}B')\boldsymbol{\mu} \equiv T\boldsymbol{\mu} \tag{14}$$

where

$$c = \text{diag}\,(\cosh \gamma_1 t, \cosh \gamma_3 t, \cdots)$$

$$s = \text{diag}\,(\sinh \gamma_1 t, \sinh \gamma_3 t, \cdots)$$

$$\gamma = \text{diag}\,(\gamma_1, \gamma_3, \cdots)$$

$$B_{ij}' = \left(\frac{4}{ad}\right) \sum_{n=1}^{\infty} \Gamma' P_{ni} P_{nj} \qquad \text{or } B = \left(\frac{4}{ad}\right) P\Gamma'\tilde{P}$$ [1]

$$\Gamma' = \text{diag}\left(\beta \frac{\tan \beta l}{\cot \beta l}, \quad \Gamma_3 \frac{\tanh \Gamma_3 l}{\coth \Gamma_3 l}, \cdots\right).$$

The series appearing in the matrix elements B_{ij}' converges, but neither uniformly everywhere nor very rapidly. Poles occur whenever $m/d \approx n/a$, i.e., higher order iris and iris guide modes "enter in resonance." This has no intrinsic physical meaning, but is a consequence of the fact that the eigenmode expansion is essentially not uniformly convergent at these points.

The transformation of the series into one which converges uniformly and more rapidly is described elsewhere [7].

The continuity equation (14) has the network interpretation of the (matrix) voltage transfer ratio of a length of transmission line terminated by a shunt admittance, which is illustrated in Fig. 4. This is indeed the relationship between the voltages at the location of the transformers in the equivalent network representation of [4]. At the same time, this shows the connection between that approach and the more general solution presented here.

We go back now to expression (9) for the driving-point admittance and compute the various constituent terms.

[1] Here, the tilde denotes transposition.

The denominator can be written as

$$\left(\frac{4}{ad}\right) \cdot \left(\frac{2}{a}\right) \left(\sum_{m=1}^{N} P_{m1} \lambda_m\right)^2 = \left(\frac{4}{ad}\right) \cdot \left(\frac{2}{a}\right) \tilde{\boldsymbol{\lambda}} P_{*1} \tilde{P}_{*1} \boldsymbol{\lambda}. \tag{15}$$

After a slight manipulation, the first term in the numerator can be written as

$$\left(\frac{2}{a\beta}\right) \tilde{\boldsymbol{\lambda}} (B + D) \boldsymbol{\lambda} \tag{16}$$

where

$$B_{ij} = \frac{4}{ad} \sum_{n=3}^{\infty} \Gamma_n P_{in} P_{jn}$$

$$D = \text{diag}\,(\gamma_1, \coth \gamma_1 t, \gamma_3 \coth \gamma_3 t, \cdots) = \gamma \cdot c \cdot s^{-1}.$$

Similarly, the second term in the numerator of (9) becomes

$$\frac{2}{a\beta} \sum \gamma_m \,\text{csch}\,(\gamma_m t) \lambda_m \mu_m = \frac{2}{a\beta} \boldsymbol{\lambda} D' \boldsymbol{\mu} \tag{17}$$

where

$$D' = \text{diag}\,(\gamma_1, \text{csch}\,\gamma_1 t, \gamma_3 \,\text{csch}\,\gamma_3 t, \cdots) = \gamma s^{-1}.$$

Collecting the expressions (15)-(17) we finally obtain

$$b_0(\boldsymbol{\lambda}) = \tilde{\boldsymbol{\lambda}} \left[B + \frac{ad}{4} (D - D'T^{-1}) \right] \boldsymbol{\lambda} \Big/ \frac{4\beta}{ad} \tilde{\boldsymbol{\lambda}} P_{*1} \tilde{P}_{*1} \boldsymbol{\lambda}$$

$$\equiv \tilde{\boldsymbol{\lambda}} U \boldsymbol{\lambda} / \tilde{\boldsymbol{\lambda}} P_{*1} \tilde{P}_{*1} \boldsymbol{\lambda}. \tag{18}$$

The matrix $P_{*1}\tilde{P}_{*1}$ is real, symmetric, and of rank one. The matrix U is real, symmetric, and nonsingular. The stationary value of the driving-point susceptance $b_0(\boldsymbol{\lambda}_0)$ is characterized by the equation

$$\frac{\partial}{\partial \lambda_i} [b_0(\boldsymbol{\lambda})(\tilde{\boldsymbol{\lambda}} P_{*1} \tilde{P}_{*1} \boldsymbol{\lambda}) - \tilde{\boldsymbol{\lambda}} U \boldsymbol{\lambda}] = 0$$

with $\partial b_0/\partial \lambda |_{\lambda=\lambda_0} = 0$, or

$$b_0 P_{*1} \tilde{P}_{*1} \boldsymbol{\lambda}_0 = U \boldsymbol{\lambda}_0. \tag{19}$$

If $z_0 = 1/b_0$ and since U is nonsingular, (19) can be reduced to the eigenvalue equation

$$U^{-1} P_{*1} \tilde{P}_{*1} \boldsymbol{\lambda}_0 = z_0 \boldsymbol{\lambda}_0. \tag{20}$$

Since $U^{-1}P_{*1}\tilde{P}_{*1}$ is of rank one, there will be only one nonzero eigenvalue satisfying (20). This is the sought stationary value of the reactance.

From the invariance of the trace of $U^{-1}P_{*1}\tilde{P}_{*1}$ it follows that the value of z_0 is given by

$$\text{tr}\,(U^{-1} P_{*1} \tilde{P}_{*1}) = \tilde{P}_{*1} U^{-1} P_{*1}. \tag{21}$$

The usefulness of this result lies in the fact that it gives an explicit value for the reactance, avoiding the calculation of modal amplitudes. If required, these can easily be computed from (20) and (14).

III. Schwinger's Representation

The aperture fields (11) have been taken as combinations of actual aperture eigenmodes.

It is interesting to compare the solution with that obtained by using the orthogonal set of functions over the aperture that was originally introduced by Schwinger [5]. For this purpose, we take $M(x)$ and $M'(x)$ such that

$$\frac{dM(x)}{dx} = F(\theta(x)) = \sum_{k=1}^{N} \rho_k \cos k\theta \tag{22a}$$

$$\frac{dM'(x)}{dx} = F'(\theta(x)) = \sum_{k=1}^{N} \sigma_k \cos k\theta \tag{22b}$$

where $\cos(\pi x/a) = \sin(\pi d/2a)\cdot\cos\theta = \alpha\cos\theta$ and the notation is similar to that used in [4]. The above definitions imply a conformal mapping of the interval $0 \le x \le a$ in the interval $a - d/2 \le x \le a + d/2$, corresponding to $0 \le \theta \le \pi$. The existence of the mapping involves better modal matching to the guide eigenmodes than that obtainable with the iris eigenmodes chosen as a basis. In order to describe the propagation on the iris, however, we shall have to go back to the latter set.

In the following, therefore, we shall need the expansions

$$\cos\frac{m\pi}{d}\left(x - \frac{a-d}{2}\right) = \sum_{l}^{\infty} A_{ml}\cos l\theta \tag{23}$$

with

$$A_{ml} = (4/\pi)\int_0^{\pi/2} \cos l\theta\cdot\cos\frac{m\pi}{d}\left(x - \frac{a-d}{2}\right) d\theta$$

and

$$\cos\frac{n\pi x}{a} = \sum_{r=1}^{n} Q_{nr}\cos r\theta \tag{24}$$

with Q_{nr} given explicitly in the Appendix. It is worth emphasizing that the existence of the *finite* expansion (24) stems from the fact that the two sets $\cos(n\pi/a)x$ and $\cos n\vartheta$ satisfy comparible boundary conditions on the iris edge. This is not the case for the sets $\sin(n\pi/a)x$ and $\sin(n\pi/d)\cdot(x-(a-d/2))$, so that the expansion of one set in terms of the other one is *intrinsically infinite*. This is borne out by the fact that the transformation matrix (13) can easily become ill-conditioned. In fact, for $m/n \approx d/a$ or, in any case, if m and n are sufficiently large, we have $m^2-(d/a)^2n^2 \approx m^2-k^2$ with k integer so that

$$P_{mn}\propto \frac{1}{m+k}\cdot\frac{1}{m-k}\,.$$

This matrix is clearly related to the matrix $H_{mk} = 1/(m+k-1)$. However, this is the so-called "Hilbert matrix," which is a well-known pathological case of ill-conditioning in numerical analysis [8].

The relationship between (11) and (22) is found from the identity

$$\begin{aligned}\frac{dM(x)}{dx} &= \sum_{k=1}^{N} \rho_k \cos k\theta \\ &\equiv \sum_m \sqrt{\frac{2}{d}}\,\frac{m\pi}{d}\lambda_m \cos\frac{m\pi}{d}\left(x-\frac{a-d}{2}\right) \\ &= \sum_{m=1}^{N}\sum_{k=1}^{\infty} \lambda_m A_{mk}\cos k\theta\cdot\frac{m\pi}{d}\sqrt{\frac{2}{d}}\end{aligned} \tag{25}$$

i.e.,

$$\boldsymbol{\varrho} = \frac{\pi}{d}\sqrt{\frac{2}{d}}\,\tilde{A}\,\Delta\boldsymbol{\lambda} \qquad \boldsymbol{\sigma} = \frac{\pi}{d}\sqrt{\frac{2}{d}}\,\tilde{A}\,\Delta\boldsymbol{\mu}$$

where $\Delta = \text{diag}\,(m)$ and $m = 1, 3, \cdots, N$.

In the new representation, (14) becomes

$$\begin{aligned}\frac{d}{a}\sum_{m=1}^{\infty}\frac{\gamma_m}{m^2}\operatorname{csch}(\gamma_m t)\sum_{k=1}^{N} A_{mk}A_{mi}\,\rho_k &= \frac{d}{a}\sum_{m=1}^{\infty}\frac{\gamma_m}{m^2}\coth(\gamma_m t)\sum_{k=1}^{N} A_{mi}A_{mk}\,\sigma_k \\ &\quad + \sum_{n=\max(i,k)}^{\infty}\frac{\Gamma_n'}{n^2}Q_{nk}Q_{ni}\sigma_k\end{aligned} \tag{26}$$

that is,

$$\frac{d}{a}\tilde{A}\Delta^{-1}D'\Delta^{-1}A\boldsymbol{\varrho} = \left(\frac{d}{a}\tilde{A}\Delta^{-1}D\Delta^{-1}A + C'\right)\boldsymbol{\sigma}$$

or

$$\frac{d}{a}E'\boldsymbol{\varrho} = \left(\frac{d}{a}E + C'\right)\boldsymbol{\sigma} \tag{27}$$

where D and D' have already been defined, E and E' are defined in (27), and

$$(C')_{ij} = \sum_{n=\max(ij)}^{\infty}\frac{\Gamma'n}{n^2}Q_{ni}Q_{nj} = (Q\Delta^{-1}\Gamma'\Delta^{-1}\tilde{Q})_{ij}. \tag{28}$$

A rearrangement of this series is given in [4, eq. (5)].

From (27), the linear relationship between the N-dimensional vectors $\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}$ is obtained:

$$\boldsymbol{\varrho} = X\boldsymbol{\sigma}. \tag{29}$$

When $N\to\infty$, $\tilde{A}\to A^{-1}$ and

$$X = \tilde{A}\Delta(c + s\gamma^{-1}B')\Delta^{-1}A = \tilde{A}\Delta T\Delta^{-1}A \tag{30}$$

is the result of a similar transformation on T. As a consequence of this approach, the matrix relationship between the $\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}$ vectors is more complicated than (14), involving the numerical summation of an additional set of infinite series. However, the series occurring in the matrix elements of C' have convergence properties superior to those of B'. Furthermore, due to the lower-triangular form of the matrix T in the expansion (24), the denominator in the expression analogous to the modified Rayleigh quotient (17) reduces to a scalar independent of $\boldsymbol{\varrho}$.

The network interpretation of the transformation (30) is to introduce at the location of the discontinuity between the guide and the iris an intermediate set of multiwinding ideal transformers.

In terms of the new representation, the expression for the driving-point susceptance becomes

$$\frac{1}{\alpha^2\beta}\tilde{\boldsymbol{\varrho}}\left[C + \left(\frac{d}{a}\right)(E - E'X^{-1})\right]a = \tilde{\boldsymbol{\varrho}}Y\boldsymbol{\varrho} = q \tag{31}$$

where $\tilde{\mathfrak{g}} = (1, \mathfrak{g}_3, \cdots) = (1, \tilde{u})$ and

$$C_{ij} = \left[\frac{1}{i} - \alpha^2 \delta_{ij}\right] \delta_{ij} + \sum_{n=\max(i,j)}^{\infty} \frac{\Gamma_n - \dfrac{n\pi}{\alpha}}{n^2} Q_{ni} Q_{nj}. \quad (32)$$

Matrix (32) represents the half of the (matrix) susceptance of an infinitely thin isolated iris. By partitioning Y as

$$\begin{Bmatrix} Y_{11} & Y_{12} \\ \tilde{Y}_{12} & Y_{22} \end{Bmatrix}$$

where Y_{11} is a number, Y_{12} is a $(1 \times N-1)$-matrix, and Y_{22} is a $(N-1) \times (N-1)$-matrix, the quadratic form (31) can be written:

$$q(\mathfrak{u}) = Y_{11} + 2 Y_{12} \mathfrak{u} + \tilde{\mathfrak{u}} Y_{22} \mathfrak{u}.$$

Its stationary value q_0, obtained when $Y_{12} = -\tilde{\mathfrak{u}}_0 Y_{22}$, is

$$q_0 = Y_{11} + Y_{12} \mathfrak{u}_0 = Y_{11} - Y_{12} Y_{22}^{-1} \tilde{Y}_{12} \quad (33)$$

As in Section II, we have been able to obtain a compact expression for the susceptance, avoiding the explicit computation of field amplitudes. If required, these can be readily obtained, in fact, $\tilde{\mathfrak{g}} = (1, -\tilde{Y}_{12} Y_{22}^{-1})$, and $\mathfrak{d}$ follows from (29).

IV. Numerical and Experimental Results

In order to compare the two methods discussed in Sections II and III and to check the results against those obtained by means of the previous approach [4], two computer programs were developed. A waveguide cavity, defined by two identical irises (two-element filter), was built as well. The cavity was gold plated in order to minimize losses and the actual measured dimensions (in centimeters) were as follows:

guide width	$a = 1.067 \pm 0.0020$
iris aperture	$d = 0.4520 \pm 0.0010$
iris thickness	$t = 0.0460 \pm 0.0020$
distance between irises	$l = 0.6535 \pm 0.0015.$

The VSWR was computed with the three methods and measured over the range 22.5–23.5 GHz, encompassing the resonance of the cavity. This occurred at 22.923 GHz.

The results are shown in Fig. 5, where the crosses are measured values.

Curve A in Fig. 5 was computed by the method previously developed and by a modal expansion of three terms per aperture; that is, four independent variables in the Rayleigh–Ritz procedure. Results obtained with this method are consistently in fairly good agreement with those obtained with the more recent approach, apart from a slight systematic deviation of the resonant frequency towards lower frequencies.

Curve B was computed with the method of Section II (fields expressed as eigenmode expansions) and a modal development of ten modes in the aperture ($N = 2 \times 10 - 1 = 19$).

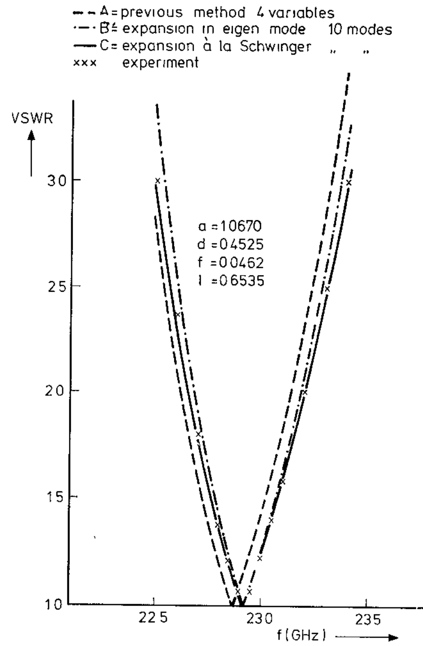

Fig. 5. Computed and experimental VSWR for two irises.

TABLE I
Resonant Frequency (GHz)

number of terms	eigenmode expansion	Schwingers' function expansion
3	22.9460	22.8621
4	22.9325	22.8873
5	22.9255	22.8991
6	22.9214	22.9073
7	22.9188	22.9133
8	22.9169	22.9167
9	22.9156	22.9200
10	22.9146	22.9228
11	22.9140	22.9245

Curve C was computed with the method of Section III (Schwinger representation and a modal development of the same order). The values of the resonant frequency as a function of the number of modes for both methods are given in Table I. It is seen that in the two methods, the limit values are approached from opposite directions.

All waveguide modes are taken into account in the matrix elements of the guide dyadic in terms of the orthogonal set on the aperture. Therefore, the solution depends only upon the order of the expansion chosen for the trial field. The computed values of the resonant frequency sensitivity for mechanical tolerances, expressed in megacycles per second/micrometers, were as follows:

$$\Delta f/\Delta a \simeq -0.7 \qquad \Delta f/\Delta d \simeq -1.5$$
$$\Delta f/\Delta l \simeq -1.7 \qquad \Delta f/\Delta t \simeq 0.3.$$

Both limit values, extrapolated from Table I, lie within the range accounted for by tolerances, the difference between them being 10 MHz.

Computation times for the three cases were in the ratio $A:B:C=20:8:3$. The reduction of computing time realized by either B or C, with respect to A, is mainly due to replacing the search for the stationary point of a nonlinear function of many variables with the closed-form expression (21) or (33) for the stationary value of the driving-point admittance.

The difference of computing time between B and C, in spite of the larger number of matrix manipulations and the summation of two sets of series involved in the latter method, is due to the better modal matching in Schwinger's representation.

A few computer results confirmed the expectation that this method becomes increasingly attractive as either the iris thickness decreases and/or the iris-to-iris distance decreases, since in either case the interaction of higher order waveguide and iris modes is enhanced.

In the limit of rather thick irises, the method of Section II appears to be competitive.

V. Conclusions

Starting from a rigorous integral-equation formulation, the variational solution was obtained to the problem of the interacting, thick inductive iris using an eigenmode expansion on the aperture, as well as an expansion à la Schwinger.

A comparison has been carried out between the two methods and the equivalent-network/variational approach presented previously. A precise two-element filter was made and tested.

Numerical results obtained with both new methods show improved agreement with the experiment.

In spite of not offering a diagonal representation for propagation in the iris, an expansion à la Schwinger appears to give definite computational advantages, especially for thin (interacting) irises. Moreover, the nonphysical situation of resonance between higher order guide and iris modes does not arise in the latter approach.

The accuracy achieved with either method is such that the uncertainty in predicting the electrical characteristics of the thick interacting iris has been reduced to that arising from the effect of mechanical tolerances.

Appendix

Recursive Formulas of the Q_{nk} Coefficients of Section III

The coefficients Q_{nk} of the expansion (26)

$$\cos\frac{n\pi x}{a} = \sum_{k=1}^{n} Q_{nk}\cos k\theta, \qquad (n, k \text{ odd})$$

with

$$\cos\frac{\pi x}{a} = \alpha\cos\theta = Q_{11}\cos\theta$$

can be obtained by means of a convenient recursive relationship derived in the following.

Since

$$\cos\frac{n\pi x}{a} = T_n\left(\cos\frac{\pi x}{a}\right) = T_n\,(\alpha\cos\theta)$$

and

$$T_n(x) = (4x^2 - 2)T_{n-2}(x) - T_{n-4}(x)$$

we set

$$\begin{aligned}\sum_{k=1}^{n} Q_{nk}\cos k\theta &\equiv (4\alpha^2\cos^2\theta - 2)\cdot\sum_{k=1}^{n-2} Q_{n-2;k}\cos k\theta - \sum_{k=1}^{n-4} Q_{n-4;k}\cos k\theta\\ &\equiv \alpha^2\sum_{k=3}^{n} Q_{n-2;k-2}\cos k\theta + \alpha^2\sum_{k=-1}^{n-4} Q_{n-2;k+2}\\ &\quad + 2(\alpha^2-1)\sum_{k=1}^{n-2} Q_{n-2;k}\cos k\theta - \sum_{k=1}^{n-4} Q_{n-4;k}\cos k\theta.\end{aligned}$$

Equating coefficients of $\cos k\theta$, we obtain the recursive formula sought:

$$Q_{nm} = \alpha^2 Q_{n-2;m-2} + \alpha^2 Q_{n-2;m+2} + 2(\alpha^2-1)Q_{n-2;m} - Q_{n-4;m}$$

with

$$Q_{nm} = 0, \qquad \text{for } m > n.$$

This form, containing only quantities of the order of unity, is more suitable for computation than an explicit polynomial form in α, involving differences of products of large numerical coefficients and large powers of α with $\alpha<1$.

A similar recursive form is easily deduced for the asymmetric case (m and n even *and* odd).

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